Feb 20.

7.2.13, 7.2.14, 7.2.16

13. Give an example of a function $f: [a,b] \to \mathbb{R}$ that is in $\mathcal{R}[c,b]$ for every $c \in (a,b)$ but which is not in $\mathcal{R}[a,b]$.

$$f(x) = \begin{cases} \dfrac{1}{(x-a)^s}, & x > a \\ 0, & x = a \end{cases} \qquad \text{where } s > 0.$$

$f(x)$ is continuous on $[c,b]$ but unbounded on $[a,b]$.

14. Suppose that $f:[a,b]\to\mathbb{R}$, that $a=c_0<c_1<\cdots<c_m=b$ and that the restrictions of f to $[c_{i-1},c_i]$ belong to $\mathcal{R}[c_{i-1},c_i]$ for $i=1,\dots,m$. Prove that $f\in\mathcal{R}[a,b]$ and that $\int_a^b f=\sum_{i=1}^{m}\int_{c_{i-1}}^{c_i} f$.

Proof. Recall Additivity theorem.

let $f:[a,b]\to\mathbb{R}$ and let $c\in(a,b)$. Then

$$f\in\mathcal{R}[a,b] \iff f\in\mathcal{R}[a,c] \text{ and } f\in\mathcal{R}[c,b].$$

In this case, $\int_a^b f=\int_a^c f+\int_c^b f$.

Prove by induction using additivity theorem.

① If $m=1$, then 14. obviously true.

② Assume $m=k>1$, 14. true,

let $m=k+1$, then by assumption $\begin{cases} f\in\mathcal{R}[a,c_k], \\ \int_a^{c_k} f=\sum_{i=1}^{k}\int_{c_{i-1}}^{c_i} f \end{cases}$

by additivity theorem,

$$f\in\mathcal{R}[a,c_k]\cap\mathcal{R}[c_k,c_{k+1}]\Rightarrow f\in\mathcal{R}[a,c_{k+1}]=\mathcal{R}[a,b],$$

and $\int_a^b f=\int_a^{c_{k+1}} f=\int_a^{c_k} f+\int_{c_k}^{c_{k+1}} f=\sum_{i=1}^{k}\int_{c_{i-1}}^{c_i} f+\int_{c_i}^{c_{k+1}} f$

$$=\sum_{i=1}^{k+1}\int_{c_{i-1}}^{c_i} f$$

∎

16. If f is continuous on $[a,b]$, $a<b$, show that there exists $c \in [a,b]$ such that we have

$$\int_a^b f = f(c)(b-a). \quad \text{(Mean Value Theorem for Integrals)}$$

Proof. f is continuous

$$\Rightarrow \exists u, \ f(u) = \max_{x \in [a,b]} f(x)$$

$$\exists v, \ f(v) = \min_{x \in [a,b]} f(x)$$

$$f(v)(b-a) = \int_a^b f(v) \le \int_a^b f \le \int_a^b f(u) = f(u)(b-a)$$

$$\Rightarrow f(v) \le \frac{\int_a^b f}{b-a} \le f(u).$$

By Intermediate Value Theorem

$$\exists c, \ f(c) = \frac{\int_a^b f}{b-a}.$$

$\blacksquare$

Beyond.

Def. (HK integrable) $f : [a,b] \to \mathbb{R}$.

$\forall \varepsilon > 0$, $\exists \delta : [a,b] \to \mathbb{R}_{>0}$, $\forall \dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$

such that $x_i - x_{i-1} \le \delta(t_i)$,

$$| S(f, \dot{P}) - L | < \varepsilon$$

then f is called HK-integrable ($\in HK[a,b]$). $(HK)\int_a^b f = L$.

Ex.

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x > 0 \\ 0 & x = 0 \end{cases}$$ is HK integrable on $[0,1]$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b] \\ 0 & x \notin \mathbb{Q} \end{cases}$$ is HK integrable on $[0,1]$.

$$R[a,b] \subset L[a,b] \subset HK[a,b]$$

introduced in MATH 4050 or MATH 5011

or defined as $\{ f \in HK[a,b] \mid |f| \in HK[a,b] \}$.

If you want to give a definition of integrable named by yourself, the definition should at least satisfy: (Axioms of integration).

⓪ Continuous functions are integrable on any $[a,b]$.

① $\int_a^b 1\,dx = b-a$

② $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$, $\int_a^b cf\,dx = c\int_a^b f\,dx$.

③ $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^a f\,dx$

④ $0 \le f \Rightarrow 0 \le \int_a^b f$.

$\forall a,b,c \in \mathbb{R}$, f,g continuous functions on $[a,b]$.

Problem*: If f is continuous, then $\int_a^b f = (R)\int_a^b f$.

the integration you defined.